# Modelling of Control Systems with Jump Signals in Orthogonal Hybrid Function Domain

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## Abstract

Jump discontinuities occurs for switching in systems of different types and such systems pose moderate to crucial difficulties in their mathematical modelling. Undesired jumps are encountered frequently in different process responses and electrical systems. For such jumps, the task of function approximation and system analysis become significantly complex in the frame work of orthogonal functions.

If such approximations are polluted with much error, then the final result is also affected and degraded. In this paper, an improved technique using the conventional orthogonal hybrid functions (HF) model has been employed to analyse and identify any system with input jump discontinuity more accurately than conventional methods using BPF or orthogonal HF set.

Results are found to be much more reliable than those obtained via conventional hybrid function analysis. The method is computationally attractive because it uses function samples as expansion coefficients, along with recursions.

***Keywords: Hybrid functions, Jump discontinuity, System modelling, Analysis, Sample-based analysis.***

## I. Introduction

Modelling of control systems had been under the scanner of researchers working in the area since about the 70's [1-3] of the last century. In the decades that followed, study of control systems spread almost exponentially. In present day research, researchers are probing the area of modelling as well as simulation more intimately so that designed systems work almost at the desired level.

For this purpose, researchers sought resort to various techniques involving non-sinusoidal orthogonal functions or piecewise constant orthogonal functions [4-6]. The reasons for such choice were:

a) These functions are staircase in nature and thus are more suitable for computerised control,
b) These functions can be integrated or differentiated by operational techniques, thus converting differential or integral equations into equivalent algebraic expressions leading to simpler algorithms and simpler solutions.
c) As a bonus, while working with these functions, we need less computer memory and the algorithms use less computational time.

Thus, piecewise constant orthogonal functions added a new dimension to system modelling. Many researchers proposed modelling of systems using these functions [4-6].

These functions started their journey with Haar functions [7, 8], followed by Rademacher functions and Walsh functions. Of all such functions, the BPF set was the most fundamental [9]. But, these functions being of staircase type, produced results in the staircase form leading to higher mean integral square error (MISE) [4, 5].

To reduce this MISE, a piecewise linear model was found more suitable to produce improved solutions.

In 1998 Deb et. al., another piecewise constant orthogonal function set, termed sample-and-hold functions [10], and was implemented to solve issues related to discrete time control systems with zero order hold.

A few years later, Deb et. al. introduced yet another set of orthogonal triangular functions [11], which provides solution in a piecewise linear pattern.

These two orthogonal function sets were linearly combined, and form an effective orthogonal function set known as 'hybrid' function (HF) set [12]. This set approximates any time function in piecewise linear manner. Also, another advantage of using this function set was, if we dropped the triangular part of the solution obtained in HF domain, we were left with the piecewise constant SHF solution [10, 12].

The name 'hybrid function' being reasonably general, we may recall that many a researcher used this name to indicate a combination of two different types of function sets used together for system analysis. For example, Ref. [13] proposed a new 'hybrid function' set with the combination of BPF and Bernoulli polynomials. In 2017, [14] proposed another 'hybrid function' set for identifying a linear multi-delay systems via a combination of BPF and Taylor's polynomials. And in 2018, he [15] came up with a different kind of 'hybrid function' set comprised of BPF and Legendre polynomials. After about a year, [16] presented yet another function set which was a combination of BPF and Chebyshev polynomials. This set of orthogonal function was applied for solving nonlinear optimal control problems having time-varying delays.

Though many such instances may be sighted for 'hybrid functions', it is clear that no one has yet used the linear combination of SHF set and TF set to obtain piecewise linear recursive solutions.

Many of the orthogonal function sets discussed above have been employed for studying different types of control systems abundantly. But none has been utilized by the researchers to approximate functions with jump discontinuities and subsequent modelling of control systems involving such discontinuities till date.

Analysis of systems having jump discontinuities in control face difficulties in their mathematical modelling. In the proposed work, it has been established that with the traditional hybrid function theory, the results obtained are not accurate. To obtain a more satisfactory as well as reliable result, an improved form of the hybrid function technique is employed to provide an improved theory. In this proposed approach, time functions involving jump discontinuities are approximated and integrated more efficiently, and reduces subsequent errors. This leads to more accurate and strongly acceptable results while dealing with the analysis of such systems.

The present method uses samples of the functions under consideration similar to conventional HF set, but introduces some special modifications to yield more accurate results. It also reduces the MISE and the mathematical burden as well.

This new algorithm can be employed in many other areas like process control and power electronics where jump situations are frequently involved.

The paper is organized in a following pattern:

(i) Brief review of hybrid functions (HF),

(ii) Approximation of function in HF domain and BPF domain are theoretically studied and compared with respect to integral square error (ISE). Also, related conditions are derived. Numerical examples are treated to complement the developed theory,

(iii) Approximation of functions with jump discontinuities using HFc, HFm and the proposed improved HF technique (HFcm) followed by an example,

(iv) Comparison of respective ISE's for approximation of jump functions, using the conventional approach (HFc) and the new modified approach (HFcm),

(v) An example is treated elaborately to compare the MISE's of HFc and HFcm approaches with BPF approximation used as the reference,

Related tables and graphs are presented to validate the proposed work.

## II. Brief Review of Hybrid Functions (HF) [12]

A linear combination of two orthogonal function sets, namely the SHF set and right-handed TF set, is known as the hybrid function (HF) set. A brief description of these two function sets is presented below:

### A. Sample-and-hold functions (SHF) [10]

A time function $f(t)$, being square integrable can be mathematically presented by one sample-and-hold function set in the semi-open interval $[0, T)$ as

$$f(t) \approx f_0 S_0 + f_1 S_1 + \cdots + f_i S_i + \cdots + f_{(m-1)} S_{(m-1)} \quad (1)$$

where, $f_i = f(ih)$, $i = 0, 1, 2, \ldots, (m-1)$

In fact, $f(ih)$ is the value of the function at $t=ih$. Thus, we call $f(ih)$ as $f_i$ andthe $f_i$'s are the expansion coefficients of $f(t)$ in sample-and-hold domain.

### B. Triangular functions (TF) [11]

We can form two sets of orthogonal triangular functions (TF), namely $\mathbf{T1}_{(m)}(t)$ and $\mathbf{T2}_{(m)}(t)$, from a conventional set of m block pulse function, i.e., $\mathbf{\Psi}_{(m)}(t)$.

$$\mathbf{\Psi}_{(m)}(t) = \mathbf{T1}_{(m)}(t) + \mathbf{T2}_{(m)}(t) \quad (2)$$

where, $\mathbf{T1}_{(m)}(t)$ is defined the *left-handed* triangular function set and $\mathbf{T2}_{(m)}(t)$ as the *right-handed* triangular function set.

In defining the hybrid function (HF) set, the *m*-set right-handed triangular function (RHTF) has been used. For convenience, in the following, we write $\mathbf{T}_{(m)}(t)$ instead of $\mathbf{T2}_{(m)}(t)$.

## III. Brief Review of Hybrid Functions (HF) [12]

For function approximation, the block pulse domain approach has the following disadvantages:

(i) the approximation is always of staircase nature, and

(ii) for computation of each block pulse function coefficients, we need one integration. So, for an $m$-term approximation of a function, we need to compute results of $m$ integrations.

For HF domain approximations of a function:

(i) the function samples serve as the expansion coefficients, and

(ii) the approximation is obtained in a piecewise-linear manner.

It is obvious that HF domain approximation will generally come up with less MISE compared to approximation in BPF domain.

In the following the integral square errors (ISE) associated with approximations in both BPF as well as HF domains, are investigated.

Let us expand the function $f(t)$ in BPF domain over $m$ sub-intervals each of interval $h$. The integral square error (ISE) for such representation is **[4]**

$$[\varepsilon]_{\mathrm{BPF}}^2 \triangleq \int_0^T \left[ f(t) - \sum_{i=0}^{m-1} c_i \psi_i(t) \right]^2 \mathrm{d}t \tag{3}$$

where, $T=mh$.

ISE in the $(i+1)$-th interval is

$$[\varepsilon_i]_{\mathrm{BPF}}^2 = \int_{ih}^{(i+1)h} \left[ f(t) - c_i \psi_i(t) \right]^2 \mathrm{d}t \tag{4}$$

Now, in the $(i+1)$-th interval, we represent $f(t)$ by its first order Taylor approximation as the linear function

$$f(t) \approx \bar{f}(t) = f(ih) + \dot{f}(ih)(t - ih) \tag{5}$$

Hence, the amplitude of the approximated function at $t = (i+1)h$ is

$$\bar{f}\left[(i+1)h\right] = f(ih) + h\dot{f}(ih)$$

Therefore, the BPF coefficient $c_i$ of $\bar{f}(t)$ is

$$c_i = \frac{f(ih) + \left\{ f(ih) + h\dot{f}(ih) \right\}}{2} = f(ih) + \frac{h}{2}\dot{f}(ih) \tag{6}$$

So, using equations (5) and (6), (4) becomes

$$[\varepsilon_i]_{\mathrm{BPF}}^2 = \int_{ih}^{(i+1)h} \left[ \left\{ f(ih) + \dot{f}(ih)(t - ih) \right\} - \left\{ f(ih) + \frac{h}{2}\dot{f}(ih) \right\} \right]^2 \mathrm{d}t$$

Thus, ISE in the $(i+1)$-th interval is

$$\mathrm{ISE}_{\mathrm{BPF}} = [\varepsilon_i]_{\mathrm{BPF}}^2 = \frac{h^3}{12}\dot{f}(ih)^2 \tag{7}$$

So, for this interval, $\mathrm{ISE}_{\mathrm{BPF}} = 0$ if $\dot{f}(ih)$ is zero.

Now, the function $f(t)$ can be expanded in HF domain over $m$ sub-intervals with sampling interval $h$. The integral square error (ISE) for this case is

$$[\varepsilon]_{\mathrm{HF}}^{2} \triangleq \int_{0}^{T}\left[ f(t) - \sum_{i=0}^{m-1} f_i\, S_i(t) - \sum_{i=0}^{m-1}\left(f_{i+1} - f_i\right) T_i(t)\right]^2 \mathrm{d}t$$

where, $f_0,\ f_1,\ f_2, \cdots,\ f_i,\ f_{i+1}, \cdots,\ f_m$ are the equidistant samples of $f(t)$ and $T = mh$.

In the $(i+1)$-th interval, hybrid function representation of the function $f(t)$ is expressed as

$$\hat{f}(t) \approx f_i + m_i\,(t - ih) \tag{8}$$

where, $$m_i \triangleq \frac{f\left[(i+1)h\right] - f(ih)}{h}, \tag{9}$$

For this case, the integral square error in the $(i+1)$-th interval is

$$[\varepsilon_i]_{\mathrm{HF}}^{2} = \int_{ih}^{(i+1)h}\left[ f(t) - \hat{f}(t)\right]^2 \mathrm{d}t$$

Using equations (5) and (8), we write

$$[\varepsilon_i]_{\mathrm{HF}}^{2} = \int_{ih}^{(i+1)h}\left[\left\{ f_i + \dot{f}(ih)(t - ih)\right\} - \left\{ f_i + m_i\,(t - ih)\right\}\right]^2 \mathrm{d}t$$

Therefore, ISE in the $(i+1)$-th interval is

$$\mathrm{ISE}_{\mathrm{HF}} = [\varepsilon_i]_{\mathrm{HF}}^{2} \quad = \frac{h^3}{3}\left(\dot{f}(ih) - m_i\right)^2 \tag{10}$$

In equation (10), $\mathrm{ISE}_{\mathrm{HF}}$ becomes zero when $\dot{f}(ih) = m_i$.

From (7) and (10) above, we have

$$\mathrm{ISE}_{\mathrm{BPF}} - \mathrm{ISE}_{\mathrm{HF}} = \frac{h^3}{12}\dot{f}(ih)^2 - \frac{h^3}{3}\left(\dot{f}(ih) - m_i\right)^2 = \frac{h^3}{12}\left(3\dot{f}(ih) - 2m_i\right)\left(2m_i - \dot{f}(ih)\right)$$

**Case I:** When $\mathrm{ISE}_{\mathrm{BPF}} - \mathrm{ISE}_{\mathrm{HF}} = 0$

Then, either $\left(3\dot{f}(ih) - 2m_i\right) = 0$ or $\left(2m_i - \dot{f}(ih)\right) = 0$

That means, the conditions are

$\dot{f}(ih) = \frac{2}{3} m_i$ and $\dot{f}(ih) = 2m_i$.

**Case II:** When $\mathrm{ISE}_{\mathrm{BPF}} - \mathrm{ISE}_{\mathrm{HF}} > 0$

$$\text{Then, both}\left(3\dot{f}(ih) - 2m_i\right) > 0 \text{ and } \left(2m_i - \dot{f}(ih)\right) > 0 \tag{11}$$

or, both $\left(3\dot{f}(ih)-2m_i\right)<0$ and $\left(2m_i-\dot{f}(ih)\right)<0$ (12)

The condition derived from (11) is

$$\frac{2}{3}m_i<\dot{f}(ih)<2m_i \tag{13}$$

This has been illustrated by Figure 1.

The condition obtained from (12) is not meaningful and hence, may be discarded.

**Case III:** When $\mathrm{ISE_{BPF}}-\mathrm{ISE_{HF}}<0$

In this case, $\left(3\dot{f}(ih)-2m_i\right)>0$ with $\left(2m_i-\dot{f}(ih)\right)<0$ (14)

or, $\left(3\dot{f}(ih)-2m_i\right)<0$ with $\left(2m_i-\dot{f}(ih)\right)>0$ (15)

The condition derived from expression (14) is

$\dot{f}(ih)\ >\ \frac{2}{3}m_i$ and $\dot{f}(ih)\ >\ 2m_i$,

which means $\dot{f}(ih)\ >\ 2m_i$ (16)

The condition derived from expression (15) is

$\dot{f}(ih)\ <\ \frac{2}{3}m_i$ and $\dot{f}(ih)\ <\ 2m_i$

Which means $\dot{f}(ih)<\frac{2}{3}m_i$ (17)

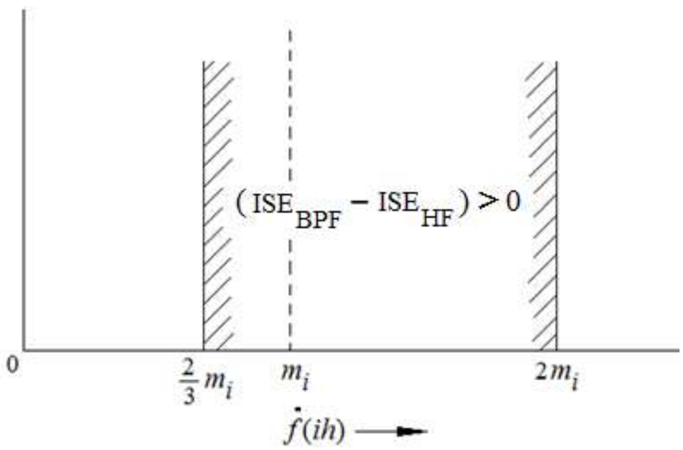


**Figure 1:** For HF domain approximation to be better than equivalent BPF based approximation, the figure shows the range (shaded portion) of $\dot{f}(ih)$ in terms of $m_i$.

### A. BPF versus HF: Comparison of Computational Burden

It is to be noted that for computation of expansion coefficients of function under study in HF domain, simply the function samples are needed and further mathematical manipulations are spared, except subtractions. That is, for HF expansion of any function in $m$ sub-intervals over a time period $T$, $(m+1)$ equidistant samples of the function are required and then we only need to compute $m$ number of subtractions with all the $2m$ coefficients.

But for BPF domain expansion of the same function in $m$ sub-intervals over a time period $T$, we need to compute $m$ number of numerical integrations over $m$ sub-intervals each of width $h$ seconds. For performing these numerical integrations, we need to consider $n$ mini-intervals within each sub-interval of width $h$. That is, in effect, we need to work with $(m \times n)$ samples and subsequently $(m \times n)$ divisions. Thus, the computational burden is increased at least $(m \times n)$ times compared to the computational effort in case of HF coefficients.

## IV. Illustrative Examples for Comparison of ISE

**Example 1:**

Consider a function $f_1(t) = \sin(\pi t)$ in the interval $t \in [0,1)$ s. We take $m$ = 8 and expand $f_1(t)$ in BPF domain as well as in HF domain.

To check whether condition (13) is satisfied, let us focus our interest in the third interval, that is, $t \in (0.25,\ 0.375)$ s, as shown in Figure 2. The slope of the function at $t$ = 0.25 is

$$\dot{f}_1(0.25) = \pi \cos(\pi t)\big|_{t=0.25} = 2.2214 \tag{18}$$

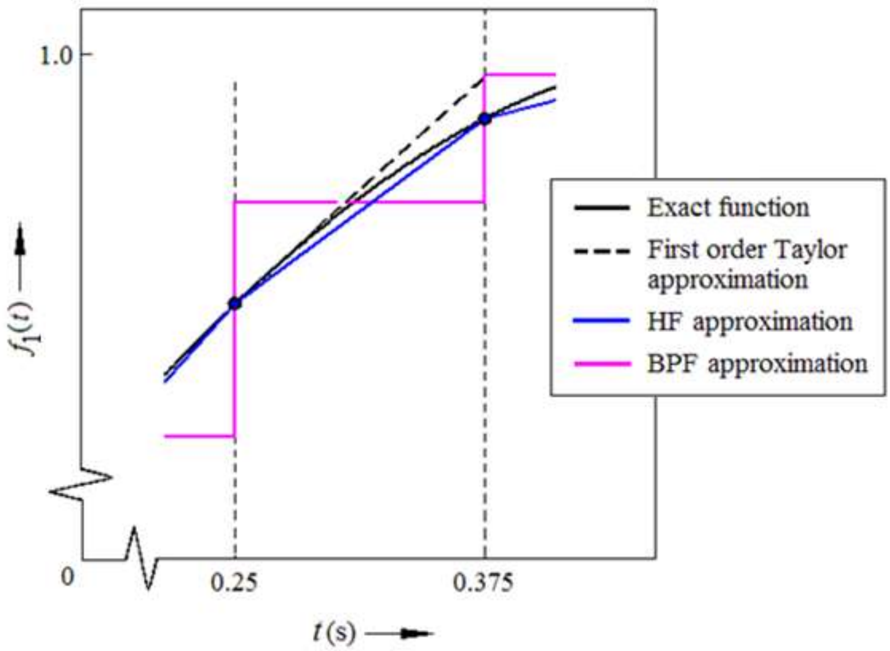


**Figure 2:** Taylor approximation of $f_1(t) = \sin(\pi t)$ along with its HF domain and BPF domain approximations in the interval (0.25, 0.375)s, referred to Figure 1.

Now we compute $m_i$ as defined by equation (9), to get

$$m_i = 1.73418$$

Thus $2m_i = 3.46836$ and $\frac{2}{3}m_i = 1.15612$ (19)

Study of (18) and (19) proves that condition (13) has been satisfied.

**Example 2:**

Similar to the last example, let us now consider the function $f_2(t) = -\sin(5\pi t)$ in the interval $t \in [0,1)$ s. We take $m$ = 8 and expand $f_2(t)$ both in BPF domain and in HF domain, as shown in Figure 3.

Now let us focus our interest in the sixth interval, that is, $t \in (0.625,\ 0.75)$ s, as shown in Figure 4. The slope of the function at $t$ = 0.625 is

$$\dot{f}_2(0.625) = -5\pi\cos(5\pi t)\big|_{t=0.625} = 14.5122 \tag{20}$$

Now we compute $m_i$ as defined by equation (9), to get

$$m_i = 2.59538 \tag{21}$$

Thus $2m_i = 5.19076$ and $\frac{2}{3}m_i = 1.73025$ (22)

Study of (20) and (22) proves that condition (16) is satisfied.

## V. Function with Jump Discontinuities and its Approximation in HF domain

Analysing systems with jump discontinuities in input functions, using any orthogonal function sets leads to an unacceptable error at the very initial stage of function approximation. This inaccuracy pollutes rest of the analysis leading to unacceptable results.

Approximation of such functions in conventional HF domain, provides piecewise linear reconstructions having approximation error within reasonable limits. But with functions having jump discontinuities, such approximations attract more error in the sub-intervals containing the jumps. This infects the final result with error.

Considering all these aspects, a modified approach was suggested by Deb *et al*. [12] termed as the 'modified' HF domain approach. To distinguish between the conventional HF domain approach and modified HF domain approach, we use the subscript 'c' for the former and 'm' for the later. That is, when any function $f(t)$ is approximated through conventional HF, we call it $\hat{f}_c(t)$, and when the function $f(t)$ is presented via the modified hybrid function method, we call it $\hat{f}_m(t)$.

In this section, we may mention yet further improved method of function approximation with jump discontinuities in HF domain. Let us approximate any function $f(t)$ approximated via this approach and call it $\hat{f}_{cm}(t)$.

If the function with jump discontinuity is a complex one, the $HF_m$ approach fails to approximate it in an error-free manner. This will be apparent from the typical function considered below

$$f_3(t) = u(t) - \exp(-3t)u(t) + Hu(t - ih) \tag{23}$$

$i$ being a positive integer.

The function is presented in Figure 5. This function may be approximated via $HF_c$ and $HF_m$ approaches, and the attempts are illustrated in Figures 6(a) and 6(b) respectively. It is observed that both the approximations contain large errors in the $i$-th sub-interval compared to other sub-intervals. To reduce such error, a further improved HF domain approach is proposed in the following.

As a first step, we decompose $f_3(t)$ as shown in Figure 7. From this figure, we realise that the function of Figure 7(a), *i.e.* $u(t)$, may be represented via $HF_c$ approach in an exact fashion. Also, the second part of this function, shown in Figure 7(b), *i.e.*, $\exp(-3t)u(t)$ may be represented via $HF_c$ approach with tolerable error. And, obviously, the remaining part shown in Figure 7(c), a delayed step function, may be approximated exactly using the $HF_m$ approach. So, we can approximate the entire function $f_3(t)$ with a combination of $HF_c$ and $HF_m$ approaches. We call this approach the $HF_{cm}$ approach.

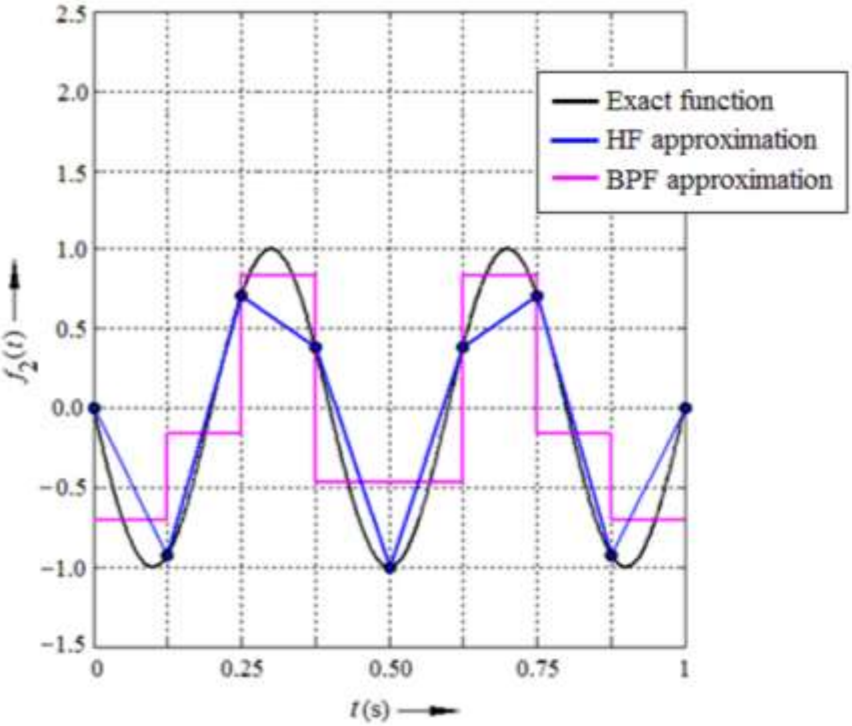


**Figure 3:** Graphical comparison of function approximation in between the HF domain and the BPF domain approach for the function $f_2(t) = -\sin(5\pi t)$, for $m = 8$ and $T = 1$ s, along with the exact plot.

We see that, $f_3(t)$ is comprised of three parts: two parts are without jump (ignoring the jump of a step function at $t = 0$), that is, $1u(t)$ and $\exp(-3t)u(t)$, and the remaining part, $Hu(t - ih)$, which involves a jump.

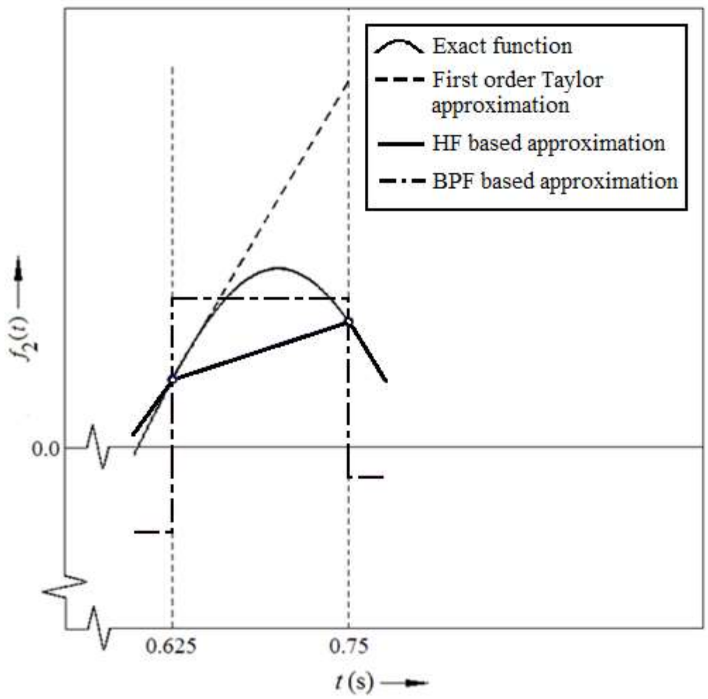


**Figure 4:** First order Taylor approximation of $f_2(t)$ = -sin($5\pi t$) along with its approximations in HF domain and BPF domain in the interval (0.625, 0.75)s (refer to Figure 3).

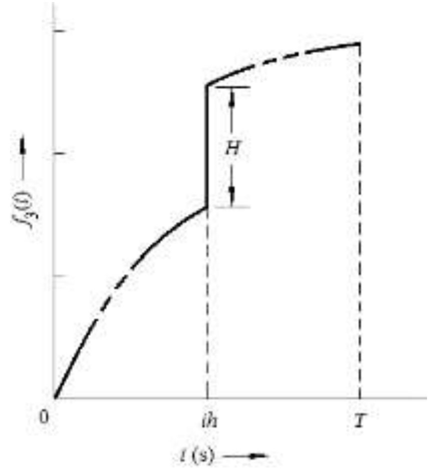


**Figure 5:** The function $f_3(t)$ with a jump discontinuity at $t = ih$.

We can now expand $f_3(t)$ in HF domain as under :

$$\begin{aligned}\hat{f}_{3,\text{cm}} &= \big[(1-1) \quad [1-\exp(-3h)] \quad [1-\exp(-6h)] \quad \cdots \quad [1-\exp\{-3(ih)\}] \quad \cdots \\ &\qquad\qquad \cdots \quad [1-\exp\{-3(m-1)h\}]\big]\mathbf{S}_{(m)} \\ &-\big[[\exp(-3h)-1] \quad [\exp(-6h)-\exp(-3h)] \quad \cdots \quad [\exp(-3ih)-\exp\{-3(i-1)h\}] \quad \cdots \\ &\qquad\qquad \cdots \quad [\exp(-3mh)-\exp\{-3(m-1)h\}]\big]\mathbf{T}_{(m)} \\ &+\big[0 \quad \cdots \quad \underset{\substack{\uparrow \\ (i+1)\text{-th term}}}{0} \quad H \quad H \quad \cdots \quad H\big]\mathbf{S}_{(m)} + \big[0 \quad 0 \quad \cdots \quad 0 \quad \cdots \quad 0\big]\mathbf{T}_{(m)} \end{aligned} \tag{24}$$

First part of (24) employs the $HF_c$ approach, while the second part uses the $HF_m$ approach. Approximation of this function $f_3(t)$ using the *new* $HF_{cm}$ approach is presented in Figure 8. We see that this approach provides almost an exact representation of $f_3(t)$ and is superior to both the $HF_c$ approach and the $HF_m$ approach. This is because $f_3(t)$ could be expressed as a combination of a jump and a non-jump functions.

However, a point should be noted that for clarity of representation, the samples appearing on the curve have not been joined by straight lines to show HF domain piecewise linear approximation.

Since we can approximate a delayed step function using the $HF_m$ approach with zero error, we can conclude that for reconstruction of equal width staircase functions, $HF_m$ approach will always be able to produce HF domain approximations *without any error*.

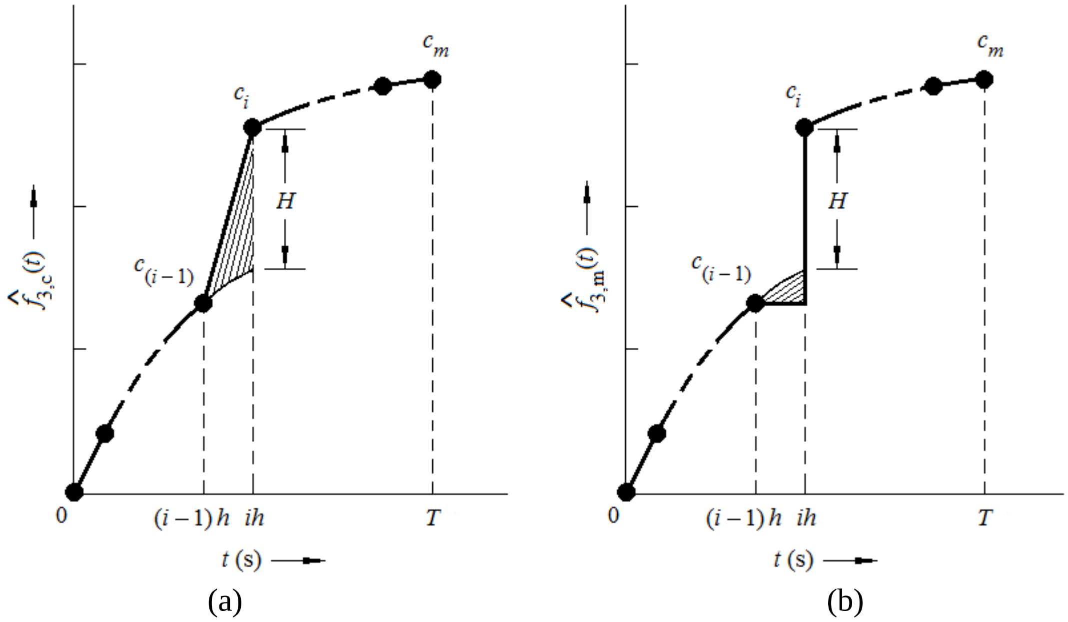


**Figure 6:** Approximation of $f_3(t)$ of Figure 5 using (a) the $HF_c$ and (b) the $HF_m$ approaches, for $m$ sub-intervals, $T$ s and $h = \frac{T}{m}$ s.

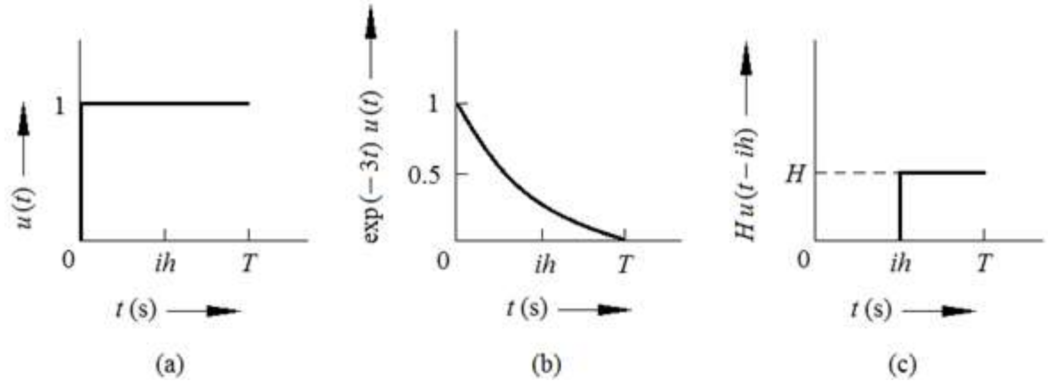


**Figure 7:** Decomposition of the function $f_3(t)$ of Figure 5.

If the function is not linear, but is a curve with one or several jumps, we can still decompose the function in different parts---the curvy part and other parts containing delayed step functions of different magnitudes. The continuous curvy part may be approximated using the traditional $\mathrm{HF_c}$ approach as in (24) above and the associated delayed step functions may be approximated using the $\mathrm{HF_m}$ approach in an *error-free* manner.

Thus, the combined approach, that is the $\mathrm{HF_{cm}}$ approach, surely will be capable to reduce the error involved in approximation further, compared to the methods, like the BPF, $\mathrm{HF_c}$ and the $\mathrm{HF_m}$ approaches.

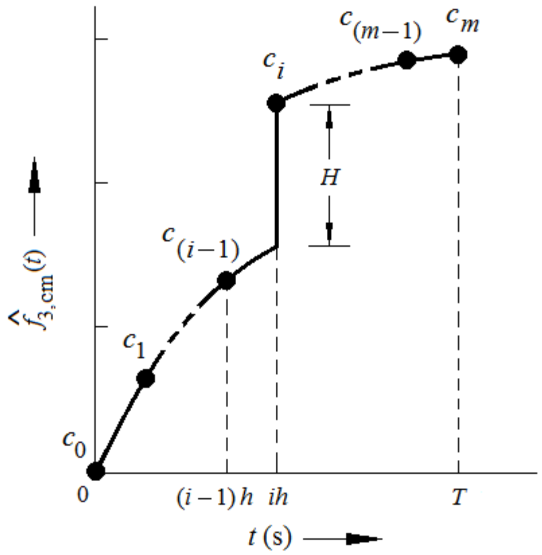


**Figure 8:** Approximation of $f_3(t)$ using the $\mathrm{HF_{cm}}$ approach.

To summarise the three techniques in general form, we consider a function $f(t)$ having several jumps in $t \in [0, T)$. The function $f(t)$ may easily be presented as a combination of a non-jump function component and $n$ (say) jump function components. That is

$$f(t) = f_{\mathrm{nj}}(t) + f_{\mathrm{j}_1}(t) + f_{\mathrm{j}_2}(t) + f_{\mathrm{j}_3}(t) + \cdots + f_{\mathrm{j}_i}(t) + \cdots + f_{\mathrm{j}_n}(t) \tag{25}$$

where $f_{\mathrm{nj}}(t)$ is the non-jump function component,

and $f_{\mathrm{j}_i}(t)$ is the $\mathrm{j}_i$-th jump function component.

$$\text{That means } f(t) = f_{\mathrm{nj}}(t) + \sum_{i=1}^{n} f_{\mathrm{j}_i}(t) \tag{26}$$

In equation (26), the function $f(t)$ has jump discontinuities at $n$ number of time instants, *i.e.*, $t = \mathrm{j}_1 h, \mathrm{j}_2 h, \ldots, \mathrm{j}_i h, \ldots, \mathrm{j}_n h,$ $i$ being a positive integer. This assures that jumps at all the instants are integral multiples of $h$.

Hence, for approximation of $f(t)$ via $\mathrm{HF_m}$ approach, we make use of the generalised **J** matrix and, following the spirit of (25), may write

$$\hat{f}(t) = \mathbf{F}_{\mathrm{S}}^{\mathrm{T}} \mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{T}}^{\mathrm{T}} \mathbf{J}_{\mathrm{j}_1, \mathrm{j}_2, \ldots, \mathrm{j}_i, \ldots, \mathrm{j}_n (m)} \mathbf{T}_{(m)} \tag{27}$$

where, $$\mathbf{J}_{\mathrm{j}_1,\mathrm{j}_2,\ldots,\mathrm{j}_i,\ldots,\mathrm{j}_n(m)} \triangleq \begin{array}{cccccccccccccccccccc} [1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1]_{(m)} \\ & & & \uparrow & & & & \uparrow & & & & \uparrow & & & & \uparrow & & & \\ & & & \mathrm{j}_1\text{-th element} & & & & \mathrm{j}_2\text{-th element} & & & & \mathrm{j}_{(n-1)}\text{-th element} & & & & \mathrm{j}_n\text{-th element} & & & \end{array}$$

and $$[a \quad b \quad c] \triangleq \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}$$

In equation (27), the entire function $f(t)$ is addressed as a whole.

Now, for the $\mathrm{HF_{cm}}$ approach, we consider equation (25) and make use of the decomposition. That is, we expand the continuous component and write

$$\hat{f}_{\mathrm{nj}}(t) \triangleq \mathbf{F}_{\mathrm{njS}}^{\mathrm{T}}\mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{njT}}^{\mathrm{T}}\mathbf{T}_{(m)} \tag{28}$$

For the jump function components of (25), we see that this part is really a sum of different delayed step functions having different amplitudes. As mentioned above, jump points of all these functions coincide with different integral multiples of $h$. Combination of these functions produces a staircase function. Such a staircase function, when expanded via $\mathrm{HF_m}$ approach, will have only the SHF component, the TF components being *zero*. Thus, the expansion of the jump function components of (25) may be written as

$$\sum_{i=1}^{n} f_{\mathrm{j}_i}(t) \triangleq \mathbf{F}_{\mathrm{jS}}^{\mathrm{T}}\mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{jT}}^{\mathrm{T}}\mathbf{T}_{(m)} \tag{29}$$

where, $\mathbf{F}_{\mathrm{jT}}^{\mathrm{T}}$ is a null matrix.

Then, using (26) and (27), complete HF domain description of $f(t)$ of (25) via $\mathrm{HF_{cm}}$ approach is given by

$$f(t) = f_{\mathrm{nj}}(t) + \sum_{i=1}^{n} f_{\mathrm{j}_i}(t)$$

or, $$\hat{f}_{\mathrm{nj}}(t) + \sum_{i=1}^{n} \hat{f}_{\mathrm{j}_i}(t) \triangleq \mathbf{F}_{\mathrm{njS}}^{\mathrm{T}}\mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{njT}}^{\mathrm{T}}\mathbf{T}_{(m)} + \mathbf{F}_{\mathrm{jS}}^{\mathrm{T}}\mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{jT}}^{\mathrm{T}}\mathbf{T}_{(m)}$$

$$= \left(\mathbf{F}_{\mathrm{njS}}^{\mathrm{T}} + \mathbf{F}_{\mathrm{jS}}^{\mathrm{T}}\right)\mathbf{S}_{(m)} + \mathbf{F}_{\mathrm{njT}}^{\mathrm{T}}\mathbf{T}_{(m)} \tag{30}$$

where, $\mathbf{F}_{\mathrm{jT}}^{\mathrm{T}}$ isa null matrix.

In the following section, function approximation of $f(t)$ with a jump interval via $\mathrm{HF_c}$, $\mathrm{HF_m}$ and $\mathrm{HF_{cm}}$ approaches, and comparison of ISE's for such approximations are taken up and discussed in detail.

## VI. Comparison of ISE's for Jump Function Approximation via HFc and HFcm Approaches: Theory

Let a function $f(t)$ with a jump discontinuity be expanded using $\mathrm{HF_c}$ approach ( $\hat{f}_{\mathrm{c}}(t)$ ) and $\mathrm{HF_{cm}}$ approach ( $\hat{f}_{\mathrm{cm}}(t)$ ). We intend to study and compare the integral square error (ISE) of these two approaches. And also, for the sake of clarity, we employ the $\mathrm{HF_m}$ ( $\hat{f}_{\mathrm{m}}(t)$ ) approach

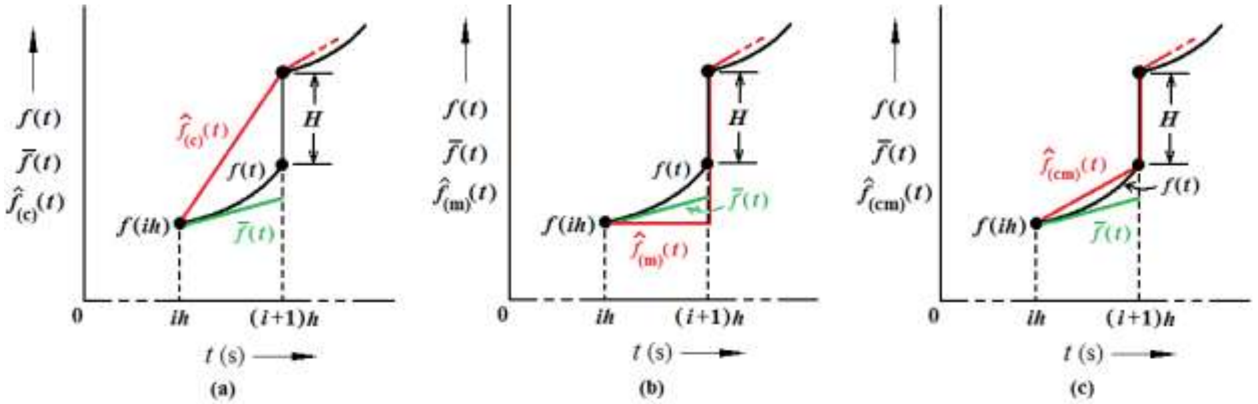


**Figure 9:** Approximation of the function $f(t)$ with the jump interval using (a) the $\mathrm{HF_c}$ approach (b) the $\mathrm{HF_m}$ approach, and (c) the $\mathrm{HF_{cm}}$ approach, for $m$ sub-intervals, $T$ s and $h = \frac{T}{m}$ s, and estimation of error where the exact function $f(t)$ has been replaced by its first order Taylor equivalent $\bar{f}(t)$.

and study the pros and cons of all the three approaches to reach some useful conclusions. These three approximations of $f(t)$ in the jump interval are illustrated in Figures 9(a), 9(b) and 9(c).

In the $(i+1)-$th jump interval, we can represent $f(t)$ by its first order Taylor approximation ( $\bar{f}(t)$ )as

$$f(t) \approx \bar{f}(t) = f(ih) + \dot{f}(ih)(t-ih) \tag{31}$$

In the $(i+1)$-th interval, $\mathrm{HF_c}$ based representation of the function $f(t)$ is expressed as

$$f(t) \approx \hat{f}_{\mathrm{c}}(t) = f(ih) + m_i'(t-ih) \tag{32}$$

where, $$m_i' \triangleq \frac{f'\left[(i+1)h\right] - f(ih)}{h}, \tag{33}$$

Referring to Figure 9(a), it is noted that at the jump point $t = (i+1)h$, ideally, we have two values of the function, $f\left[(i+1)h\right]$ and $f'\left[(i+1)h\right]$, where

$$f'\left[(i+1)h\right] = f\left[(i+1)h\right] + H$$

where, $H$ is the jump amplitude at $t = (i+1)h$.

Thus, we may have two possible slopes in HF domain reconstruction, given by (9) and (33).

For this case, the ISE in the $(i+1)$-th jump interval, from (31) and (32), is

$$[\varepsilon_i]^2_{\mathrm{HFc}} \triangleq \int_{ih}^{(i+1)h} \left[f(t) - \hat{f}_{\mathrm{c}}(t)\right]^2 \mathrm{d}t$$

$$= \frac{h^3}{3}\left(\dot{f}(ih) - m_i'\right)^2 \tag{34}$$

From equations (9) and (33), we can write

$$m_i' = m_i + \frac{H}{h} \tag{35}$$

Therefore, using relation (35), (34) can be written as

$$\mathrm{ISE}_{\mathrm{HFc}} = \frac{h^3}{3}\left(\dot{f}(ih) - m_i - \frac{H}{h}\right)^2 \tag{36}$$

Now, we consider the $\mathrm{HF_m}$ approach.

With this approach, the function is reconstructed in the jump interval via a SHF having an amplitude *f*(*ih*). Comparing this approximation with the first order Taylor equivalent of *f*(*t*) given by (31), we express the ISE as

$$[\varepsilon_i]^2_{\mathrm{HFm}} = \int_{ih}^{(i+1)h} \left[\left\{f(ih) + \dot{f}(ih)(t - ih)\right\} - \left\{f(ih)\right\}\right]^2 \mathrm{d}t$$

$$\text{or,}\ \ \mathrm{ISE}_{\mathrm{HFm}} = \frac{h^3}{3}\left(\dot{f}(ih)\right)^2 \tag{37}$$

Comparing (36) and (37), we can write

$$\mathrm{ISE}_{\mathrm{HFc}} = \mathrm{ISE}_{\mathrm{HFm}} + \frac{h^3}{3} m_i' \left\{m_i' - 2\dot{f}(ih)\right\} \tag{38}$$

Using equation (38), we can study the different cases depending upon the nature of the function. Figure 10 illustrates a typical function and its approximations in the jump interval using first order Taylor series, $\mathrm{HF_c}$ and $\mathrm{HF_m}$ approaches.

In the following, we study different cases of approximation and related integral square errors.

**Case I:** In (38), when $\mathrm{ISE}_{\mathrm{HFc}} = \mathrm{ISE}_{\mathrm{HFm}}$

$$h\left\{m_i' - 2\dot{f}(ih)\right\} = 0\,. \tag{39}$$

Thus, we have, $m_i' = 2\dot{f}(ih)$

**Case II:** When $\mathrm{ISE}_{\mathrm{HFc}} < \mathrm{ISE}_{\mathrm{HFm}}$

$m_i' < 0$ and $h\left\{m_i' - 2\dot{f}(ih)\right\} > 0$, or vice versa. (40)

Hence, for this case, either $m_i' < 0$ and $m_i' > 2\dot{f}(ih)$

or, $m_i' > 0$ and $m_i' < 2\dot{f}(ih)$

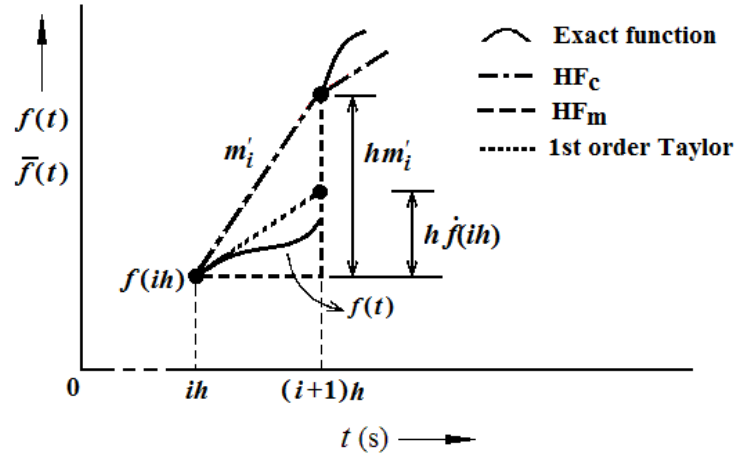


**Figure 10:** A typical function $f(t)$ and its approximations in the jump interval using first order Taylor series, $\mathrm{HF_c}$ and $\mathrm{HF_m}$ approaches.

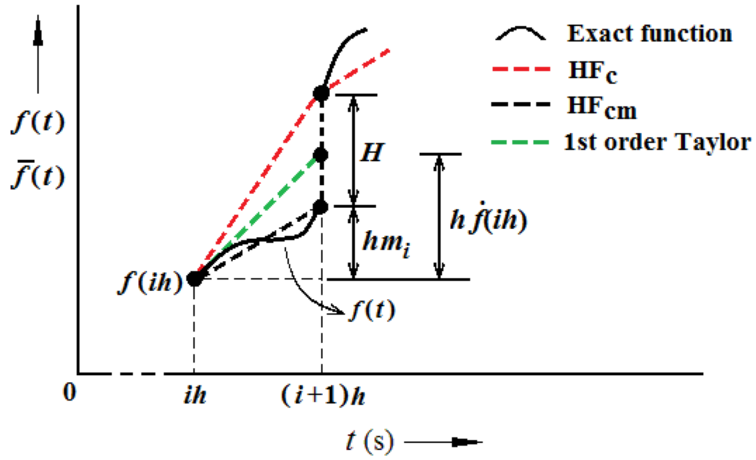


**Figure 11:** A typical function $f(t)$ and its approximations in the jump interval using first order Taylor series, $\mathrm{HF_c}$ and $\mathrm{HF_{cm}}$ approaches.

**Case III:** When $\mathrm{ISE_{HFc}} > \mathrm{ISE_{HFm}}$

$$m_i' > 0 \text{ and } h\left\{m_i' - 2\dot{f}(ih)\right\} > 0 \,. \tag{41}$$

Here we have, either $m_i' > 0$ and $m_i' > 2\dot{f}(ih)$

or, $m_i' < 0$ and $m_i' < 2\dot{f}(ih)$.

If we consider the $\mathrm{HF_{cm}}$ approach for approximating the function, the ISE in the $(i+1)$-th jump interval can be mathematically expressed as

$$\mathrm{ISE_{HFcm}} = \frac{h^3}{3}\left(\dot{f}(ih) - m_i\right)^2 \tag{42}$$

Therefore from (36) and (42), we can write the relationship between two ISEs as

$$\mathrm{ISE_{HFc}} = \mathrm{ISE_{HFcm}} + \left[\frac{H^2 h}{3} - \frac{2}{3}h^2 H\left\{\dot{f}(ih) - m_i\right\}\right] \tag{43}$$

From equation (43), we can study the following cases.

**Case I:** When $\mathrm{ISE}_{\mathrm{HFc}} = \mathrm{ISE}_{\mathrm{HFcm}}$

$$\left[\frac{H^2h}{3} - \frac{2}{3}h^2H\left\{\overset{\square}{f}(ih) - m_i\right\}\right] = 0$$

$$\text{or,}\quad h\left\{\overset{\square}{f}(ih) - m_i\right\} = \frac{H}{2} \tag{44}$$

which implies, $\overset{\square}{f}(ih) > m_i$ ,considering $H$ to be positive.

**Case II:** When $\mathrm{ISE}_{\mathrm{HFc}} > \mathrm{ISE}_{\mathrm{HFcm}}$

$$h\left\{\overset{\square}{f}(ih) - m_i\right\} < \frac{H}{2} \tag{45}$$

**Case III:** When $\mathrm{ISE}_{\mathrm{HFc}} < \mathrm{ISE}_{\mathrm{HFcm}}$

$$h\left\{\overset{\square}{f}(ih) - m_i\right\} > \frac{H}{2} \tag{46}$$

As in the case of condition (44), condition (46) also implies that $\overset{\square}{f}(ih) > m_i$, considering $H$ to be positive.

In Figure 11, a typical function $f(t)$ and its approximations in the jump interval using first order Taylor series, $\mathrm{HF_c}$ and $\mathrm{HF_{cm}}$ approaches have been shown.

## VII. Comparison of MISE's for Jump Function Approximations via HFc and HFcm Approaches: Numerical Example

In Figures 6(a) the $\mathrm{HF_c}$ method reconstructs the function $f_3(t)$, while in Figure 6(b), the $\mathrm{HF_m}$ approach approximates the function. Study of these two approximations shows that though for many cases the $\mathrm{HF_m}$ approach is superior to $\mathrm{HF_c}$ approach, its superiority cannot be ensured in a general manner. This has also been proved through equations (39), (40) and (41). But when we employ the $\mathrm{HF_{cm}}$ approach, $f_3(t)$ is reconstructed in Figure 8, thereby reducing the approximation error which is apparent from inspection of the Figures 6 and 8.

Now we take up a numerical example to establish the issue.

**Example 3:**

Let us consider the function $f_3(t)$ of Figure 5, having a jump at $t = ih = 0.5$ s (say) and the jump $H = 0.5$. Calling this function $f_4(t)$, we can write

$$f_4(t) = \begin{cases} 1 - \exp(-3t) & \text{for } t \le 0.5 \\ 1.5 - \exp(-3t) & \text{for } t > 0.5 \end{cases} \tag{47}$$

This function is approximated via the $HF_c$ approach using the spirit of equation (28) and via the $HF_{cm}$ approach using the spirit of equation (30).

The function $f_4(t)$ may be decomposed as

$$f_4(t) = u(t) - \exp(-3t)u(t) + 0.5u(t-0.5) \tag{48}$$

This decomposition helps to show that the $HF_{cm}$ approach reconstructs the function making the approximation error a minimum.

Now we consider the function $f_4(t)$ and approximate the same via the $HF_{cm}$ approach.

It is clear from (48) that the function contains a delayed unit step function as a component. We now go for a combination by expressing the non-jump component functions by $HF_c$ approach and the jump component function via $HF_m$ approach following equation (30). That is, for $m$ = 10 and $T$ = 1 s, we express the component functions of $f_4(t)$ in HF domain as

$$\begin{aligned}
f_{4,\mathrm{nj}}(t) &= u(t) - \exp(-3t)u(t) \\
&\approx [0.00000000 \quad 0.25918178 \quad 0.45118836 \quad 0.59343034 \quad 0.69880578 \\
&\qquad 0.77686984 \quad 0.83470111 \quad 0.87754357 \quad 0.90928204 \quad 0.93279449]\mathbf{S}_{(10)} \\
&\quad + [0.25918178 \quad 0.19200658 \quad 0.14224198 \quad 0.10537545 \quad 0.07806405 \\
&\qquad 0.05783127 \quad 0.04284246 \quad 0.03173847 \quad 0.02351244 \quad 0.01741844]\mathbf{T}_{(10)}
\end{aligned}$$

$$\begin{aligned}
f_{4,\mathrm{j}}(t) &= 0.5u(t-0.5) = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5 \quad 0.5]\mathbf{S}_{(10)} \\
&\quad + [0 \quad 0 \quad 0 \quad 0 \quad 0.5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{J}_{5(10)}\mathbf{T}_{(10)}
\end{aligned}$$

Following equation (30), we write

$$\begin{aligned}
f_4(t) &= f_{4,\mathrm{nj}}(t) + f_{4,\mathrm{j}}(t) \\
&\approx [0.00000000 \quad 0.25918178 \quad 0.45118836 \quad 0.59343034 \quad 0.69880578 \\
&\qquad 1.27686984 \quad 1.33470111 \quad 1.37754357 \quad 1.40928204 \quad 1.43279449]\mathbf{S}_{(10)} \\
&\quad + [0.25918178 \quad 0.19200658 \quad 0.14224198 \quad 0.10537545 \quad 0.07806405 \\
&\qquad 0.05783127 \quad 0.04284246 \quad 0.03173847 \quad 0.02351244 \quad 0.01741844]\mathbf{T}_{(10)} \\
&\quad + [0 \quad 0 \quad 0 \quad 0 \quad 0.5 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\mathbf{J}_{5(10)}\mathbf{T}_{(10)}
\end{aligned}$$

Thus, the above function $f_4(t)$ has been approximated using the improved HF domain approach in the best possible way.

Table 1 tabulates the MISE's of approximation of $f_4(t)$ via block pulse function approach along with the $HF_c$ and the $HF_{cm}$ approaches for different sets of $m$. It has also been noted that MISE for the $HF_{cm}$ approach is less than the BPF domain approach and the $HF_c$ approach. Thus, the approximation via the $HF_{cm}$ approach proves to be the best for all values of $m$.

Figure 12 depicts the variation of MISE's with the number of sub-intervals *m* using the BPF, $HF_c$ and $HF_{cm}$ approaches.

The comparison of two different HF domain approximations with respective BPF approximation will be clearer if we define two ratios of MISE's as

$$\frac{MISE_{BPF}}{MISE_{HF_c}} \triangleq R_{BPF\text{-}HF_c}$$

and $$\frac{MISE_{BPF}}{MISE_{HF_{cm}}} \triangleq R_{BPF\text{-}HF_{cm}}$$

All the ratios are computed and tabulated in last two columns of Table 1, where the ratios speak for themselves. It is noted that $R_{BPF\text{-}HF_c}$ is always less than one. This is contrary to our general expectation. And that is why, cell-wise MISE for the three approximations, namely, BPF based approximation and two HF based approximations---$HF_c$, and $HF_{cm}$ has been investigated for $m$= 10 and $T$= 1 s. Those results are presented in Table 2.

Further, with increasing value of *m*, say from 10 to 100 or even to still higher values, it is noted from Table 1 that the $HF_{cm}$ method is always more accurate, with respect to MISE, compared to BPF or $HF_c$ based approximations.

**Table 1:** HF domain approximations of the function $f_4(t)$ of Example 3 and its comparison of MISE's using the $HF_c$ and the $HF_{cm}$ approaches along with BPF domain approximation, for increasing values of $m$ with $T$ = 1 s.

| Number of sub-intervals used ($m$) | MISE using BPF approach ($MISE_{BPF}$) | MISE using $HF_c$ approach ($MISE_c$) | MISE using $HF_{cm}$ approach ($MISE_{cm}$) | $\frac{MISE_{BPF}}{MISE_{HF_c}} \triangleq R_{BPF\text{-}HF_c}$ | $\frac{MISE_{BPF}}{MISE_{HF_{cm}}} \triangleq R_{BPF\text{-}HF_{cm}}$ |
|---|---|---|---|---|---|
| 10 | 0.00123578 | 0.00824798 | 0.00001111 | 0.14982820 | 111.23132313 |
| 20 | 0.00031103 | 0.00415614 | 0.00000070 | 0.07483627 | 444.32857143 |
| 30 | 0.00013841 | 0.00277467 | 0.00000014 | 0.04988341 | 988.64285714 |
| 40 | 0.00007789 | 0.00208202 | 0.00000004 | 0.03741078 | 1947.25000000 |
| 50 | 0.00004986 | 0.00166600 | 0.00000002 | 0.02992797 | 2493.00000000 |
| 60 | 0.00003463 | 0.00138850 | 0.00000000 | 0.02494058 | Inf |
| 70 | 0.00002544 | 0.00119023 | 0.00000000 | 0.02137402 | Inf |
| 80 | 0.00001948 | 0.00104150 | 0.00000000 | 0.01870379 | Inf |
| 90 | 0.00001539 | 0.00092581 | 0.00000000 | 0.01662328 | Inf |
| 100 | 0.00001247 | 0.00083325 | 0.00000000 | 0.01496550 | Inf |

From Table 2 it is noticed that for the interval immediately before the jump instant, MISE is maximum, which is 0.08237329 for $HF_c$ and for $HF_{cm}$ minimum and is only 0.00000456. For the same interval, with approximation in BPF domain, MISE is 0.00050707. Whereas for all other cells, $HF_c$ and $HF_{cm}$ methods have the same MISE and its magnitude is much less than that of BPF method. Aso, the sum of MISEs of all the ten intervals for $HF_c$ or $HF_{cm}$ methods, excluding the interval just before the jump instant, is only 0.00010650 and the same is 0.01185074 for the BPF method. This detailed study proves the efficiency of HF based approximation and indicates its superiority over the BPF technique.

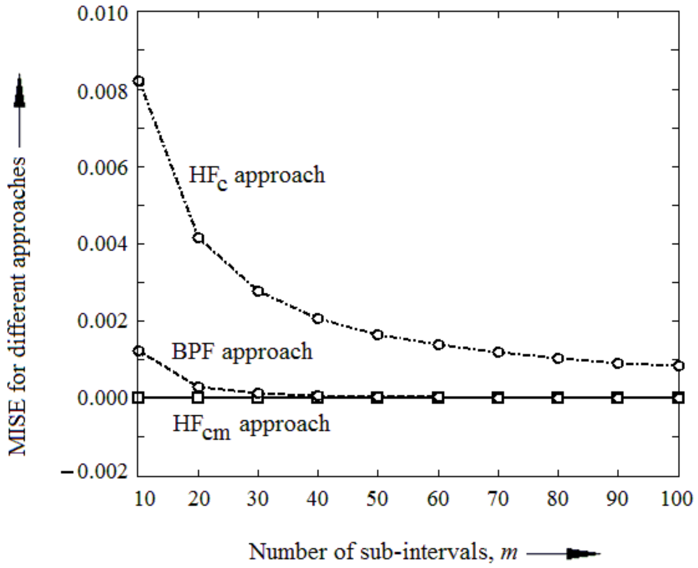


**Figure 12:** Approximation of function of Example 3, with $T$ = 1 s, and comparison of MISEs for increasing values of $m$, with the BPF, $HF_c$ and $HF_{cm}$ approaches.

**Table 2:** Approximations of the function $f_4(t)$ of **Example 3** for $m$ = 10 and $T$ = 1 s, and its cell-wise comparison of MISEs with BPF, $HF_c$ and $HF_{cm}$ based approaches.

| Sub-interval no. for $m$ =10, $h$= 0.1 s $T$= 1 s | Segment-wise MISE using BPF approach | Segment-wise MISE using $HF_c$ approach | Segment-wise MISE using $HF_{cm}$ approach |
|---|---|---|---|
| 1 | 0.00558955 | 0.00005024 | 0.00005024 |
| 2 | 0.00306761 | 0.00002757 | 0.00002757 |
| 3 | 0.00168354 | 0.00001513 | 0.00001513 |
| 4 | 0.00092395 | 0.00000830 | 0.00000830 |
| 5 | **0.00050707** | **0.08237329** | **0.00000456** |
| 6 | 0.00027829 | 0.00000250 | 0.00000250 |
| 7 | 0.00015273 | 0.00000137 | 0.00000137 |
| 8 | 0.00008382 | 0.00000075 | 0.00000075 |
| 9 | 0.00004600 | 0.00000041 | 0.00000041 |
| 10 | 0.00002525 | 0.00000023 | 0.00000023 |

## VIII. Conclusion

In this paper, we have utilised the hybrid function domain to present a HF-based improved approach to approximate time functions involving *jump discontinuities*, with improved accuracy. Compared to approximation in traditional hybrid function domain, or orthogonal triangular function and block pulse function domain approximations, this modified approach has amply reduced the mean integral square error (MISE). This improvement has been tested with a few numerical examples.

When handling functions with jump discontinuities, *e.g.*, their approximation or integration, this paper has proved the superiority of the HF based *improved* technique over HF based *conventional* approach analytically.

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